decomposition of the injectant. Injection was initiated one second after motor ignition and the motor burned for a total of 4.1 (±0.1) sec. Upon opening the motor case, it was found that all the propellant had burned except for some random patches of thickness no greater than about $\frac{1}{32}$ in. In tests using the high flow rate, upon commencing injection, the yellow-white plume immediately changed to a solid black plume of roughly the same dimensions as the plume before injection and stayed this way until burnout. Injection was initiated 1.5 sec after ignition and the motor burned for a total of 4.2 (±0.05) sec. The propellant residues were similar to those found in the low flow rate tests. Note that in both tests burning was not extinguished, but the total burning time was increased somewhat.

The foregoing tests indicate that, if solid motor thrust termination is to be achieved by injecting an extinguishant onto the burning surface, a concerted development program is needed. This should involve a coordinated program of research on extinguishants and on propellant additives which would serve to increase the effectiveness of specific extinguishants that would be used.

References

[1] "Freon," Dupont Tech. Bull. no. B-4B (1962).

[2] Levy, A., et al., "Inhibition of lean methane flames," *Eighth Symposium on Combustion* (Williams and Wilkins, Baltimore, Md., 1962), p. 524.

Matrix Derivation of a Short-Term Linear Rendezvous Equation

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TO optimize any system, one needs its equations of motion. If these equations of motion can be represented as a "linear plant," then optimization is relatively easy. By a "linear plant," one means that the equations of motion are in the form $\dot{x} = Fx + Gu$, where F and G are matrices that can be functions of time, x is the state vector, and u is the control vector.

Although the rendezvous problem can be linearized in rectangular coordinates, the problem is linearized better in a rotating coordinate system. For rendezvous with an object in a circular orbit and a rotating coordinate system, F is a constant matrix. For noncircular orbital rendezvous, the time-varying parts of F can be viewed as perturbations.

To obtain the equations of motion in a rotating coordinate system, first consider the matrix equation

$$x_r = Ax_s \tag{1}$$

where A is an orthogonal transformation matrix, x_s is a vector that represents a position in inertial space, and x_r is a vector that represents the same position in a rotated coordinate system. Now let A be a function of time, so that x_r is a function of time which represents a position x_s in a rotating coordinate system.

The matrix A has an inverse so that

$$x_s = A^{-1}(t)x_r \tag{2}$$

Differentiate twice with respect to time:

$$\dot{x}_s = \dot{A}^{-1}x_r + A^{-1}\dot{x}_r \tag{3}$$

$$\ddot{x}_s = \ddot{A}^{-1}x_r + 2\dot{A}^{-1}\dot{x}_r + A^{-1}\ddot{x}_r \tag{4}$$

By Newton's laws,

$$\ddot{x}_s = f_s/m \tag{5}$$

where f_s is the force acting on a body of mass m. To write the force in the rotating coordinate system, f_r, one must matrix-multiply Eq. (5) by A:

$$f_r/m = A(f_s/m) = A\ddot{x}_s \tag{6}$$

Using Eq. (4),

$$\ddot{x}_r = (f_r/m) - 2A\dot{A}^{-1}\dot{x}_r - A\ddot{A}^{-1}x_r \tag{7}$$

The last two terms in Eq. (7) can be considered as "inertial" forces. The $-2A\dot{A}^{-1}\dot{x}_r$ is the generalized coriolis force, and the $-A\ddot{A}^{-1}x_r$ is the generalized centrifugal force.

For a frame rotating at a constant speed, these generalized inertial forces are identical to the classical ones. Let the frame rotate at a constant speed about the z axis of the inertial frame; then

$$A = \begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

Let ω be a vector whose magnitude defines the rotation rate and whose direction defines the axis of rotation. Then

$$-2A\ddot{A}^{-1}\dot{x}_r = -2\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\dot{x}_r = -2(\omega \times \dot{x}_r) \tag{9}$$

which is the usual way of representing the coriolis force. Now

$$-A\ddot{A}^{-1}x_r = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}x_r = -\omega \times (\omega \times x_r) \tag{10}$$

where the cross product is defined in terms matrix multiplication. This is the usual representation of centrifugal force.

Having obtained the generalized inertial force, the usual orbital equation can be obtained from Eq. (7). Fixing the axis of rotation in the z direction and replacing ωt by θ, Eq. (7) becomes

$$\ddot{x}_r + 2\begin{pmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\dot{x}_r + \begin{pmatrix} -\dot{\theta}^2 & -\ddot{\theta} & 0 \\ \ddot{\theta} & -\dot{\theta}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}x_r = \frac{f_r}{m} \tag{11}$$

Examine the equations of the individual components. Let

$$X_r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

If one requires that the inertial values of z and $\dot{z}$ be zero and that the z component of f_r for $z = 0$ be zero, then $z(t) = 0$. Let $y = r$, require $x(t) = 0$, and let f_r be a central force:

$$_r = \begin{pmatrix} 0 \\ f_r \\ 0 \end{pmatrix}$$

Then the equation for the x component is

$$-2\dot{\theta}\dot{r} - \ddot{\theta}r = 0 \tag{12}$$

The equation for the r component or y is

$$\ddot{r} - \dot{\theta}^2 r = f_r/m \tag{13}$$

Equations (12) and (13) are the usual orbit equations.

To find the linear rendezvous equations easily, one uses the usual linearization technique. Define

$$x_\Delta = X_r - x_k \tag{14}$$

where X_r is a general solution in the rotating coordinates of Eq. (11), and x_k is a specific known solution. One expands

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Eq. (11) as a Taylor series in x_Δ and neglects all higher-order terms. Define x_k and the rotating coordinate system such that $x = z = 0$, i.e.,

$$x_k = \begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix}$$

The f_r then is written as

$$f_r = \begin{pmatrix} \dfrac{-\mu x}{(x^2 + y^2 + z^2)^{3/2}} \\ \dfrac{-\mu y}{(x^2 + y^2 + z^2)^{3/2}} \\ \dfrac{-\mu z}{(x^2 + y^2 + z^2)^{3/2}} \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{-\mu}{r^2} \\ 0 \end{pmatrix} + \begin{pmatrix} \dfrac{-\mu}{r^3} x_\Delta \\ \dfrac{2\mu}{r^3} y_\Delta \\ \dfrac{-\mu}{r^3} z_\Delta \end{pmatrix} + \text{higher terms} \qquad (15)$$

One rewrites Eq. (11) as an equation in z_Δ and a two-dimensional matrix equation:

$$\ddot{z}_\Delta + (\mu/r^3) z_\Delta = 0 \qquad (16)$$

$$\begin{pmatrix} \ddot{x}_\Delta \\ \ddot{y}_\Delta \end{pmatrix} + 2\begin{pmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{pmatrix}\begin{pmatrix} \dot{x}_\Delta \\ \dot{y}_\Delta \end{pmatrix} + \begin{pmatrix} -\dot{\theta}^2 & -\ddot{\theta} \\ +\ddot{\theta} & -\dot{\theta}^2 \end{pmatrix}\begin{pmatrix} x_\Delta \\ y_\Delta \end{pmatrix} = \begin{pmatrix} -(\mu/r^3)x_\Delta \\ 2(\mu/r^3)y_\Delta \end{pmatrix} \qquad (17)$$

The solution in the z_Δ coordinate is independent of that in the x_Δ-y_Δ plane.

For a coordinate system of a circular orbiting body, $\theta = \omega t$, $\dot{\theta} = \omega$, $\ddot{\theta} = 0$, $\mu/r^3 = \omega^2$; then from Eq. (17), one gets

$$\begin{pmatrix} \ddot{x}_\Delta \\ \ddot{y}_\Delta \end{pmatrix} + 2\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} \dot{x}_\Delta \\ \dot{y}_\Delta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -3\omega^2 \end{pmatrix}\begin{pmatrix} x_\Delta \\ y_\Delta \end{pmatrix} = 0 \qquad (18)$$

For a general orbiting coordinate system, θ is a function of time. Equation (16) can be solved explicitly. The solution of Eq. (17) would be obtained by the Piano-Baker method. This solution would be similar to that of Eq. (18). To solve Eq. (18), one must define four-dimensional state vector and its differential equation.

Now let y be the state vector whose components are

$$y_1 = x_\Delta \qquad y_3 = y_\Delta$$
$$y_2 = \dot{x}_\Delta \qquad y_4 = \dot{y}_\Delta$$

Then Eq. (18) becomes

$$\dot{y} = By \qquad (19)$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & +3\omega^2 & 0 \end{pmatrix}$$

In order to solve Eq. (19) easily, one should put B into a form similar to the Jordan normal form. This is an easy process if one uses the Cayley-Hamilton theorem. If one defines a new state vector z such that $y = Qz$, where

$$Q = \begin{pmatrix} -2 & -2 & 1 & 0 \\ 2j\omega & -2j\omega & 0 & \omega \\ j & -j & 0 & \frac{2}{3} \\ \omega & \omega & 0 & 0 \end{pmatrix} \qquad (20)$$

or $z = Q^{-1}y$, where

$$Q^{-1} = \begin{pmatrix} 0 & -j/\omega & \frac{3}{2}j & 1/2\omega \\ 0 & j/\omega & -\frac{3}{2}j & 1/2\omega \\ 1 & 0 & 0 & 2/\omega \\ 0 & -3/\omega & 6 & 0 \end{pmatrix} \qquad (21)$$

Then Eq. (19) becomes

$$\dot{z} = Cz = \begin{pmatrix} -j\omega & 0 & 0 & 0 \\ 0 & j\omega & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 \end{pmatrix} z \qquad (22)$$

whose solution is (initial conditions denoted by subscript 0)

$$z_1 = e^{-j\omega t} z_{10} \qquad z_3 = \omega t\, z_{40} + z_{30}$$
$$z_2 = e^{+j\omega t} z_{20} \qquad z_4 = z_{40} \qquad (23)$$

The initial z is obtained from the initial y by Eq. (21). To obtain the solution in terms of real y, use Eq. (20).

It is very important to note that the solution has only short-term applicability. It is applicable for rendezvous that occurs in much less than one orbit. To verify that the solution has only short-term applicability, one looks at the linear time-varying part of y_1 or x_Δ. To be applicable, z_{40} must be proportioned to a change in the semimajor axis a_Δ, or the change in the mean angular rate, $\omega_\Delta = -3\omega a_\Delta/2a$, of the rendezvousing vehicle, where

$$a_\Delta = -(2y_2/\omega) + 2y_3 \qquad (24)$$

The negative sign on the y_2 is due to the direction of coordinate rotation in relation to the orbiting vehicle. The coordinate system is rotating counter-clockwise, so that the front of the orbiting vehicle looks in the negative y_1 direction.

An arbitrary thrust vector in the y coordinate system is

$$u = \begin{pmatrix} 0 \\ u_2(t) \\ 0 \\ u_4(t) \end{pmatrix}$$

where $u_2(t)$ and $u_4(t)$ are arbitrary functions of time. This is written then in z coordinates.

The solution for arbitrary thrust can be obtained easily. Consider the matrix equation

$$\dot{Z} = CZ \qquad (25)$$

where $Z_0 = I$, and I is the identity matrix. Each column of Z is a solution to Eq. (22). For this problem,

$$Z(t) = \begin{pmatrix} e^{-j\omega t} & 0 & 0 & 0 \\ 0 & e^{+j\omega t} & 0 & 0 \\ 0 & 0 & 1 & \omega t \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (26)$$

This is called the "fundamental matrix" or "matrizant." Consider

$$(d/dt)Z^{-1}Z = \dot{Z}^{-1}Z + Z^{-1}CZ = 0 \qquad (27)$$

Therefore

$$\dot{Z}^{-1} = -Z^{-1}C \qquad (28)$$

Consider

$$(d/dt)Z^{-1}z = -Z^{-1}Cz + Z^{-1}(Cz + u)$$

which can be written as

$$z(t) = Z(t)Z_0^{-1}z_0 + Z(t)\int_{t_0}^{t} Z^{-1}(t)\, u(t)dt$$

To put results into a real form, transform back to the y coordinate system. It is desirable in optimization problems to remove the imaginary quantities in Eq. (22). This can be done by a very simple matrix transformation.